Kerr-Gauss-Bonnet Black Holes: An Analytical Approximation

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Gauss-Bonnet gravity provides one of the most promising frameworks to study curvature corrections to the Einstein action in supersymmetric string theories, while avoiding ghosts and keeping second order field equations. Although Schwarzschild-type solutions for Gauss-Bonnet black holes have been known for long, the Kerr-Gauss-Bonnet metric is missing. In this paper, a five dimensional Gauss-Bonnet approximation is analytically derived for spinning black holes and the related thermodynamical properties are briefly outlined.

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I. INTRODUCTION

In any attempt to perturbatively quantize gravity as a field theory, higher-derivative interactions must be included in the action. Such terms also arise in the effective low-energy action of string theories. Furthermore, higher-derivative gravity theories are intrinsically attractive as in many cases they display features of renormalizability and asymptotic freedom. Among such approaches, Lovelock gravity [1] is especially interesting as the resulting equations of motion contain no more than second derivatives of the metric, include the self interaction of gravitation, and are free of ghosts when expanding around flat space. The four-derivative Gauss-Bonnet term is most probably the dominant correction to the Einstein-Hilbert action [2] when considering the dimensionally extended Euler densities used in the Lovelock Lagrangian which straightforwardly generalizes the Einstein approach in (4+n)-dimensions. The action therefore reads as:

$$\begin{aligned} S_{GB} &= \frac{1}{16\pi G}\int d^D x\sqrt{-g}\Big[-2\Lambda + R \\ &+ \alpha(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2)\Big], \end{aligned} \tag{1}$$

where α is a coupling constant of dimension (length)^2, and G the $D-$dimensional Newton's constant defined as $G = 1/M_*^{D-2}$ in terms of the fundamental Planck scale M_*. Gauss-Bonnet gravity was shown to exhibit a very rich phenomenology in cosmology (see, *e.g.*, [3] and references therein), high-energy physics (see, *e.g.*, [4] and references therein) and black hole theory (see, *e.g.*, [5] and references therein). It also provides interesting solutions to the dark energy problem [6], offers a promising framework for inflation [7, 8], allows useful modification of the Randall-Sundrum model [9] and, of course, solves most divergences associated with the endpoint of the Hawking evaporation process [10].

Either in D-dimensions or in 4-dimensions with a dilatonic coupling (required to make the Gauss-Bonnet term dynamical), Gauss-Bonnet black holes and their rich thermodynamical properties [11] have only been studied in the non-spinning (*i.e.* Schwarzschild-like) case. Although some general features can be derived in this framework, it remains mostly unrealistic as both astrophysical black holes and microscopic black holes possibly formed at colliders [12, 13, 14] are expected to be rotating (*i.e.* Kerr-like). Of course, the latter – which should be copiously produced at the *Large Hadron Collider* if the Planck scale is in the TeV range as predicted by some large extra-dimension models [15]– are especially interesting for Gauss-Bonnet gravity as they could be observed in the high-curvature region of General Relativity and allow a direct measurement of the related coupling constant [4]. The range of impact param-

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eters corresponding to the formation of a non-rotating black hole being of zero measure, the Schwarzschild or Schwarzschild-Gauss-Bonnet solutions are mostly irrelevant. This is also of experimental importance as only a few quanta should be emitted by those light black holes, evading the Gibbons [16] and Page [17] arguments usually pointed out to neglect the angular momentum of primordial black holes.

It should be underlined that D-dimensional spinning black hole solutions are anyway very important within different theoretical frameworks (*e.g.*, in conservation law studies) [18]. Thanks to perturbation theory, several attempts were made [19] to derive the solution. In the following, we focus on an analytical approach.

II. 5D SOLUTION

To investigate the detailed properties of black holes in Lovelock gravity, it is therefore mandatory to derive the general solution, *i.e* the metric for the spinning case. Unlike the numerical attempts that were presented in [20] for degenerated angular momenta, the present paper focuses on the exact solution in 5 dimensions.

Einstein equations in Gauss-Bonnet gravity with a cosmological constant Λ read as

$$\begin{aligned}
&R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \Lambda g_{\mu\nu} \\
&+ \ \alpha\Bigg[\frac{1}{2}g_{\mu\nu}\Big(R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2\Big) \\
&- \ 2RR_{\mu\nu} + 4R_{\mu\gamma}R^{\gamma}_{\nu} + 4R_{\gamma\delta}R^{\gamma\delta}_{\mu\nu} \\
&- \ 2R_{\mu\gamma\delta\lambda}R^{\gamma\delta\lambda}_{\nu}\Bigg],
\end{aligned} \tag{2}$$

and the 5-dimensional metric in the spherically-symmertic Kerr-Schild type can be written as

$$\begin{aligned}
ds^2 &= dt^2 - dr^2 - (r^2+a^2)\sin^2\theta d\phi_1^2 \\
&- (r^2+b^2)\cos^2\theta d\phi_2^2 - \rho^2 d\theta^2 \\
&- 2dr\left(a\sin^2\theta d\phi_1 + b\cos^2\theta d\phi_2\right) \\
&- \beta\Bigg(dt - dr - a\sin^2\theta d\phi_1 \\
&- b\cos^2\theta d\phi_2\Bigg)^2,
\end{aligned} \tag{3}$$

where $\rho^2 = r^2+a^2\cos^2\theta+b^2\sin^2\theta$ and unknown function $\beta = \beta(r,\theta)$.

The $\theta\theta$ component of Einstein equations reads :

$$A\beta'' + B\beta'^2 + C\beta' + D\beta + E = 0, \tag{4}$$

where

$$\begin{aligned}
A &= r\rho^2(4\alpha\beta - \rho^2) \\
B &= 4\alpha r\rho^2 \\
C &= 2\Big[4\alpha\beta(\rho^2 - r^2) - \rho^2(\rho^2 + r^2)\Big] \\
D &= 2r(2r^2 - 3\rho^2) \\
E &= 2r\Lambda\rho^4.
\end{aligned}$$

This equation can be split into 2 relations depending respectively only upon β and $z \equiv \beta\beta'$ as independent unknown functions. It is then possible to introduce a new function $f(r,c)$ where $c = a^2\cos^2\theta + b^2\sin^2\theta$ so that the equations are equivalent to the following system:

$$\begin{aligned}
&\beta'' + 2\Big(\frac{\rho^2+r^2}{r\rho^2}\beta' - \frac{2r^2-3\rho^2}{\rho^4}\beta - \Lambda\Big) \\
&- \ \frac{f(r,c)}{r\rho^4} = 0
\end{aligned} \tag{5}$$

$$z' + 2\frac{\rho^2-r^2}{r\rho^2}z - \frac{1}{2}\frac{f(r,c)}{\alpha r\rho^2} = 0. \tag{6}$$

Introducing the new function $p(r,c)$ via the transformation

$$f(r,c) = \frac{\rho^4}{r}\frac{\partial p(r,c)}{\partial r}, \tag{7}$$

the second equation can be solved (with $p_r \equiv \partial p(r,c)/\partial r$), leading to :

$$z = \frac{1}{2}\frac{(\int p_r dr + 2C_{21}\alpha)(r^2+c^2)}{\alpha r^2} = (\beta\beta'), \tag{8}$$

where C_{ij} are constants of integration in the i-th equation. This equation can be integrated to obtain:

$$\beta^2 = \frac{1}{\alpha}\int\left(p\frac{r^2+c^2}{r^2}\right)dr + 2C_{21}\frac{r^2-c^2}{r} + C_{20}. \tag{9}$$

The first equation results in

$$\begin{aligned}
\beta &= \Bigg(C_{12}r - C_{11}(r^2-c^2) \\
&\quad - \ r\int\frac{(p_r - 2r^2\Lambda)(r^2-c^2)}{r}dr \\
&\quad + \ (r^2+c^2)\int(p_r + 2\Lambda r^2)dr\Bigg)\frac{1}{r(r^2+c^2)}
\end{aligned} \tag{10}$$

where a simple integration by parts

$$\int p_r\frac{r^2-c^2}{r}dr = p\frac{r^2-c^2}{r} - \int p\frac{r^2+c^2}{r^2}dr \tag{11}$$

leads to:

$$\begin{aligned}
&\beta r(r^2+c^2) = c_{12}r + C_{11}(r^2-c^2) \\
&+ \ r\int\left(p\frac{r^2+c^2}{r^2}\right)dr + \frac{\Lambda r^3}{6}(r^2+c^2).
\end{aligned} \tag{12}$$

As the same integral combination

$$Q = \int \left(p \frac{r^2 + c^2}{r^2} \right) dr. \tag{13}$$

is involved, the system leads to the quadratic equation:

$$\begin{aligned} &\alpha\beta^2 - (r^2 + c^2)\beta \\ + &\left(C_{32} + C_{31} \frac{r^2 - c^2}{r} \right. \\ + &\left. \frac{\Lambda r^2}{6}(r^2 + 2c^2) \right) = 0 \end{aligned} \tag{14}$$

where C_{3i} are new integration constants obtained from a combination of C_{2i} and C_{1i}.

Taking into account the asymptotes at infinity (and therefore finding the values of the integration constants, M being the ADM mass), this leads to

$$\beta = \frac{\rho^2 \pm \sqrt{\rho^4 - 4\alpha M - \frac{2}{3}\alpha\Lambda r^2(2\rho^2 - r^2)}}{2\alpha} \tag{15}$$

where the "-" branch should be chosen so as to recover the usual Kerr solution in the limit $\alpha \to 0$. In case of a vanishing rotation ($a = b = 0$) the obtained solution corresponds to one suggested in Ref. [21]. When used in the metric (3), this leads to the exact Kerr - Gauss - Bonnet - (anti) - deSitter solutions of Einstein equations. As only the $\theta\theta$ component of the field equations was used to derive this result, the compatibility with the other components was carefully checked. Although the equations are far too intricate to allow for analytical investigations, numerical results show that they are indeed fulfilled.

III. TRANSFORMATION TO THE BOYER - LINGUIST FORM

To obtain the value of horizon radius r_h it is necessary to transform the metric (3) back to the Boyer-Linguist form with:

$$\begin{aligned} dt' &= Adt + Bdr + Cd\theta, \\ d\phi_1' &= Dd\phi_1 + Edr + Fd\theta, \\ d\phi_2' &= Dd\phi_2 + Hdr + Fd\theta. \end{aligned}$$

Taking into account that the processes relevant for thermodynamical investigations take place in the surroundings of the horizon, M/ρ^2 can be considered as a small parameter and used for a Taylor expansion of β as

$$\beta \approx \frac{M}{\rho^2} + \frac{8M^2\alpha}{\rho^6}.$$

The Boyer - Linguist parameterization imposes, as a necessary condition, vanishing coefficients for non-diagonal components except for $dtd\phi_1$ and $dtd\phi_2$. This leads to a system of 8 equations with 8 variables. The solutions are explicitly the Boyer-Linguist parameterization of the Kerr-Gauss-Bonnet metric.

Solving those equations (without substituting the direct expression of $\rho(r, \theta)$), one obtains that:

i) all the coefficients before the components $d\theta dx$, x being an arbitrary coordinate vanish automatically, as in the classical Kerr case ;

ii) The coefficients A and D could be set equal to 1 to recover the classical case;

iii) Other coefficients are:

$$\begin{aligned} B &= B_1/B_2 \\ E &= E_1/E_2 \\ H &= H_1/H_2 \end{aligned}$$

where

$$\begin{aligned} B_1 &= -\rho^6 a(r^2 + b^2) \\ B_2 &= +r^4\rho^6 + 8M^2\alpha a^2 \cos^2\theta r^2 - M\rho^4 a^2 \cos^2\theta r^2 \\ &\quad +\rho^6 b^2 r^2 + \rho^6 b^2 a^2 - M\rho^4 r^4 \\ &\quad +r^2\rho^6 a^2 + 8M^2\alpha r^4 - M\rho^4 b^2 r^2 \\ &\quad +8M^2\alpha b^2 r^2 + M\rho^4 b^2 \cos^2\theta r^2 \\ &\quad -8M^2\alpha b^2 \cos^2\theta r^2, \\ E_1 &= -(a^2 + r^2)\rho^6 b \\ E_2 &= r^4\rho^6 + 8M^2\alpha a^2 \cos^2\theta r^2 - M\rho^4 a^2 \cos^2\theta r^2 \\ &\quad +\rho^6 b^2 r^2 + \rho^6 b^2 a^2 - M\rho^4 r^4 \\ &\quad +r^2\rho^6 a^2 + 8M^2\alpha r^4 - M\rho^4 b^2 r^2 \\ &\quad +8M^2\alpha b^2 r^2 + M\rho^4 b^2 cos^2\theta r^2 \\ &\quad -8M^2\alpha b^2 \cos^2\theta r^2, \\ H_1 &= +Mr^2(8M\alpha a^2 \cos^2\theta - \rho^4 a^2 \cos^2\theta - \rho^4 r^2 \\ &\quad +8M\alpha r^2 - \rho^4 b^2 + 8M\alpha b^2 \\ &\quad +\rho^4 b^2 \cos^2\theta - 8M\alpha b^2 \cos^2\theta) \\ H_2 &= +r^4\rho^6 + 8M^2\alpha a^2 \cos^2\theta r^2 - M\rho^4 a^2 \cos^2\theta r^2 \\ &\quad +\rho^6 b^2 r^2 + \rho^6 b^2 a^2 - M\rho^4 r^4 \\ &\quad +r^2\rho^6 a^2 + 8M^2\alpha r^4 - M\rho^4 b^2 r^2 \\ &\quad +8M^2\alpha b^2 r^2 + M\rho^4 b^2 cos^2\theta r^2 \\ &\quad -8M^2\alpha b^2 \cos^2\theta r^2. \end{aligned}$$

After substituting those coefficients in the metric (3) and some algebra, the metric becomes:

$$\begin{aligned} &ds^2 = \\ + &dt^2 - (r^2 + a^2)\cos^2\theta d\phi_1^2 - (r^2 + b^2)\sin^2\theta d\phi_2^2 \\ - &\rho^2 d\theta^2 - \left(\frac{M}{\rho^2} + \frac{8M^2\alpha}{\rho^6} \right) \left(dt \right. \\ + &\left. a\sin^2\theta d\phi_1 + b\cos^2\theta d\phi_2 \right)^2 \\ - &\frac{\Phi}{\rho^6 \left((r^2 + a^2)(r^2 + b^2) - r^2\rho^2 (\frac{M}{\rho^2} + \frac{8M^2\alpha}{\rho^6}) \right)} dr^2 \end{aligned} \tag{16}$$

where Φ is a coefficient whose value is irrelevant as this investigation requires only the denominator of the last term (g_{11} component of the metric), which is:

$$\rho^6\left((r^2+a^2)(r^2+b^2)-r^2\rho^2\beta\right) \tag{17}$$

IV. THERMODYNAMICAL PROPERTIES

For the investigation of the black hole topology, one has to study the singular points of the metric component g_{11}, *i.e.* study the zeros of the expression (17):

$$(r^2+a^2)(r^2+b^2)-r^2\rho^2\beta=0. \tag{18}$$

This is an 8th order equation in ρ when β is Taylor expanded at the lowest order in α. As show in [25], the cosmological constant can change the temperature. In the following, we restrict our study to the $\Lambda=0$ case. Using the value of β given in (15), one obtains:

$$M=\frac{M^*}{4\alpha r^4\rho^4} \tag{19}$$

where

$$\begin{aligned} M^* \ &= \ r_+^4\rho^8-4\alpha(r_+^2+a^2)^2(r_+^2+b^2)^2 \\ &\quad +4\alpha r_+^2\rho^4(r_+^2+a^2)(r_+^2+b^2)+r_+^8\rho^4 \end{aligned}$$

and r_+ is horizon radius. When $\alpha\to 0$ this leads to the usual Kerr case.

It should be underlined that the angle variable θ is included in the expression (19) (as $\rho^2=r_+^2+a^2\cos^2\theta+b^2\sin^2\theta$), which indicates a good choice of coordinates. To remove this dependence one has to set $\theta=\pi/4$. This allows computing the temperature, which requires the surface gravity given by:

$$\kappa^2=-\frac{1}{4}g^{tt}g^{ij}(\partial_i g_{tt})(\partial_j g_{tt})|_{r=r_+}. \tag{20}$$

In the considered case, this leads to

$$\kappa=-\frac{1}{4}(1+\beta)[g^{rr}(\partial_r\beta)^2+g^{\theta\theta}(\partial_\theta\beta)^2]|_{r=r_+}. \tag{21}$$

After the substituting all the values, this formula becomes

$$\kappa=-\frac{1}{4}(1+\beta)\left(\frac{\kappa_1}{\kappa_2}-\frac{(\frac{\partial}{\partial\theta}\beta)^2}{\rho^2}\right), \tag{22}$$

where

$$\begin{aligned} \kappa_1 \ &= \ (\beta r^2\cos^2\theta(a^2-b^2)-(r^2+a^2)(r^2+b^2) \\ &\quad +\beta r^2(r^2+b^2))\left(\frac{\partial}{\partial r}\beta\right)^2 \\ \kappa_2 \ &= \ -\cos^2\theta(a^2-b^2)+(r^2+b^2). \end{aligned}$$

The black hole temperature T can be easily computed by $T=\kappa/2\pi$. The pure Kerr 5D formula, as given in [26], leads to:

$$T=\frac{r_+^2\Delta'}{4\pi(r_+^2+a^2)(r_+^2+b^2)}, \tag{23}$$

where $\Delta=(r_+^2+a^2)(r_+^2+b^2)/r_+^2$.

Fig.1 displays the pure Kerr temperature and the Kerr-Gauss-Bonnet temperature. As expected, both values become very close for large masses. They differer by about 5% in the limit of very small masses for $\alpha=1$ in Planck units.

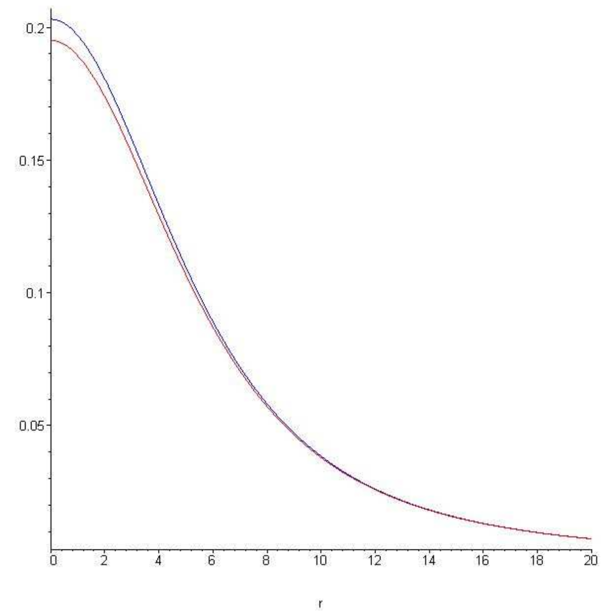

FIG. 1: Black hole temperature T (y-axis, relative Plank values) versus black hole size r_+ (x-axis, relative Plank values) for pure Kerr case (lower line) and Kerr-Gauss-Bonnet case (upper line).

V. DISCUSSION AND CONCLUSIONS

If, as suggested by geometrical arguments and by low-energy effective superstring theories, Gauss-Bonnet gravity is a realistic path toward a full quantum theory of gravity, then Kerr-Gauss-Bonnet black holes are probably among the most important objects to understand the physical basis of our World. This article has established the solution of Einstein equations in the 5-dimensional Gauss-Bonnet theory. This allows investigating into the details the physics of "realistic" spinning black holes, both from a pure theoretical and from a phenomenological (in the framework of low Planck-scale models) viewpoint.

Some improvements and developments can be foreseen. First, it should be very welcome to obtain the same kind

of solutions for any number of dimensions. Unfortunately the method introduced in this article is not easy to generalize and a specific study should be made for each case. Then, it would be interesting to compute the greybody factors for those black holes. Following the techniques of [23], it is possible (although not straightforward) to obtain a numerical solution as soon as the metric is known, at least in the $\Lambda = 0$ case. The Kerr-Gauss-Bonnet-(Anti)-deSitter situation is more intricate as the metric is nowhere flat, requiring a more detailed investigation, as suggested in [24].

VI. ADDENDUM

After the completion of this work, it was pointed out by N. Deruelle that some analytical checks (*e.g.* on the trace of the field equations) with Mathematica suggest that this solution is not exact. It could mean that the Kerr-Gauss-Bonnet solution should be looked for in a wider class of solutions than Kerr-Schild ones. Nevertheless, our results remain "heuristically" relevant as an approximation taking into account the accurate numerical checks.

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